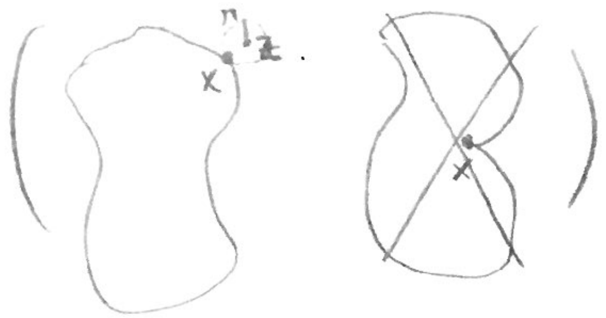
Existencia para Dirichlet

Teorema 1: Sea $U \subseteq \mathbb{R}^n$ abierto y acotado y ∂U cumpliendo la condición de la esfera exterior (i.e. $\forall x \in \partial U, \exists r>0$ y $z \in \mathbb{R}^n$ / $\overline{B_r(z)} \cap \overline{U} = \{x\}$). Si $g \in C^0(\partial U) \Rightarrow$ el problema



$$\begin{cases} \Delta u = 0 & \text{en } U \\ u|_{\partial U} = g \end{cases}$$

tiene solución en $C^2(U) \cap C^0(\overline{U})$.

(Ver Gilbarg-Trudinger, 23-27)

OBS: ¿Por qué para usar ref. de variable pedimos $g \in C^0$ y $g' \in C$?

Existencia para Dirichlet/Poisson

- Soluc. fundamental de la ec. de Laplace ($\mathbb{R}^n$ con $n>1$)

Buscamos u / $\Delta u = 0$ en $\mathbb{R}^n$ y $u(x) = \tilde{u}(\|x\|)$. Definiendo

$$r = \|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \Rightarrow \partial u/\partial x_i = \frac{d}{dr}\tilde{u}(r)\, \partial r/\partial x_i = \tilde{u}'\, x_i/r$$

$$\Rightarrow \partial^2 u/\partial x_i^2 = \left(\tilde{u}'/r\right)' x_i^2/r + \tilde{u}'/r \;\therefore\; \Delta u = r\left(\tilde{u}'/r\right)' + n\,\tilde{u}'/r =$$

$$= \tilde{u}'' + (n-1)\tilde{u}'/r \rightsquigarrow \text{Ec. de Euler} \rightsquigarrow \tilde{u}(r) = \begin{cases} a \ln r + b, & n=2 \\ a/r^{n-2} + b, & n>2 \end{cases}$$

$$\Rightarrow u(x) = \begin{cases} a \ln\|x\| + b, & n=2 \\ a/\|x\|^{n-2} + b, & n>2 \end{cases}$$

<u>Definición</u>: Se llama <u>solución fundamental de la ec. de Laplace</u> a la función $\Phi : \mathbb{R}^n - \{0\} \to \mathbb{R}$ /

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln(\|x\|), & n=2 \\ \left((n-2)\,\omega_n \|x\|^{n-2}\right)^{-1}, & n \geq 3 \end{cases}$$

Con $\omega_n = \int_{\partial B_1(0)} dS = 2\pi^{n/2} / \Gamma(n/2)$.

<u>OBS</u>: Si $n=3$, $\Phi(x) \sim 1/\|x\|$ $\equiv$ potencial de una carga puntual en $x=0$.

<u>Teorema 2</u>: Dados $U \subseteq \mathbb{R}^n$ abierto y acotado, y $f : \bar{U} \to \mathbb{R}$ / $f \in C^1(U) \cap C^0(\bar{U})$, la función $u : \bar{U} \to \mathbb{R}$ / $\boxed{u(x) = \lim_{\delta \to 0} \int_{U - B_\delta(x)} \Phi(x-y) f(y)\, dV_y}$ pertenece a $C^2(U) \cap C^0(\bar{U})$ y cumple $\Delta u = -f$ en U.

(int. impropia)

(Ver JOHN, 151-158; GILBARG-TRUDINGER, 54-56.)

D// "<u>Simplificación</u>": $f \in C^3(\mathbb{R}^n)$, $\text{sop} f = \overline{\{x \in \mathbb{R}^n / f(x) \neq 0\}} \subseteq U$.

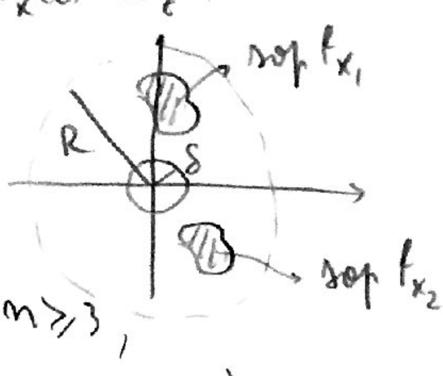
Es claro que:

$$\int_{U - B_\delta(x)} \Phi(x-y) f(y)\, dV_y = \int_{\text{sop} f - B_\delta(x)} \Phi(x-y) f(y)\, dV_y = \int_{\mathbb{R}^n - B_\delta(x)} \Phi(x-y) f(y)\, dV_y$$

Con el cambio de variable $y \mapsto z = x - y$ se tiene que

$$u(x) = \lim_{\delta \to 0} \int_{\mathbb{R}^n - B_\delta(0)} \Phi(z) \underbrace{f(x-z)}_{\equiv f_x(z)} dV_z \underset{(*)}{=} \lim_{\delta \to 0} \int_{B_R(0) - B_\delta(0)} \Phi(z) f_x(z)\, dV_z$$

(*) $\exists R > 0$ / $\text{sop} f_x \subseteq B_R(0)$, $\forall x \in U$.



Como Φ es integrable en toda bola (e.g., si $n \geq 3$,

$$\int_{B_R(0)} \Phi(z)\, dV \sim \int_{B_R(0)} \frac{1}{r^{n-2}} r^{n-1}\, dr\, d\Theta = \int_{B_R(0)} r\, dr\, d\Theta \Big)$$

y además $f \in C^3(\mathbb{R}^n)$, se puede ver que:

③

$$\Delta u(x) = \lim_{\delta\to 0} \int_{B_R(0)-B_\delta(0)} \Phi(z)\, \underbrace{\Delta_x f(x-z)}_{\Delta_z f(x-z) \,=\, \Delta f_x(z)}\, dV_z = \lim_{\delta\to 0} \int_{B_R(0)-B_\delta(0)} \Phi(z)\, \Delta f_x(z)\, dV_z$$

En resumen:

$$\boxed{\Delta u(x) = \lim_{\delta\to 0} \int_{B_R(0)-B_\delta(0)} \Phi(z)\, \Delta f_x(z)\, dV_z}$$

Apliquemos ahora la id.: $\int_\Omega (u\Delta v - v\Delta u)\, dV = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \hat{n}} - v \frac{\partial u}{\partial \hat{n}}\right) dS$

para $u = \Phi$, $v = f_x$, $\Omega = B_R(0) - B_\delta(0)$. Usando que $\Delta\Phi = 0$ en tal Ω $\Rightarrow$ $\Delta u(x) = \lim_{\delta\to 0} \int_{\partial(B_R(0)-B_\delta(0))} \left(\Phi \frac{\partial f_x}{\partial \hat{n}} - f_x \frac{\partial \Phi}{\partial \hat{n}}\right) dS$, ∂

$\partial(B_R(0)-B_\delta(0)) \to \partial B_R(0) \cup \partial B_\delta^-(0)$

Como $\frac{\partial f_x}{\partial \hat{n}} \wedge f_x \equiv 0$ en $\partial B_R(0)$ $\Rightarrow$

$$\Delta u(x) = -\lim_{\delta\to 0} \int_{\partial B_\delta(0)} \Big(\underbrace{\Phi \frac{\partial f_x}{\partial \hat{n}}}_{①} - \underbrace{f_x \frac{\partial \Phi}{\partial \hat{n}}}_{②}\Big) dS$$

① $\left| \int_{\partial B_\delta(0)} \Phi \underbrace{\frac{\partial f_x}{\partial \hat{n}}}_{\hat{n}\cdot\nabla f_x} dS \right| \leq \max_{\mathbb{R}^n} \|\nabla f\| \cdot \int_{\partial B_\delta(0)} |\Phi|\, dS \xrightarrow[\delta\to 0]{} 0$

$\int_{\partial B_\delta(0)} |\Phi|\, dS$: $\ln\delta/_{2\pi} \cdot 2\pi\delta$, si $n=2$; $\frac{1}{(n-2)w_n \delta^{n-2}} \cdot w_n \delta^{n-1}$, si $n \geq 3$

② $\frac{\partial\Phi}{\partial\hat{n}}(z) = \hat{n}\cdot\nabla\Phi(z) = \frac{z}{\|z\|} \cdot \frac{-z}{w_n\|z\|^n} = \frac{-1}{w_n\|z\|^{n-1}}$ $\Rightarrow$

$\Rightarrow \int_{\partial B_\delta(0)} f_x \frac{\partial\Phi}{\partial\hat{n}}\, dS = \int_{\partial B_\delta(0)} f_x \left(\frac{-1}{w_n \delta^{n-1}}\right) dS = \frac{-1}{\int_{\partial B_\delta(0)} dS} \int_{\partial B_\delta(0)} f_x\, dS \xrightarrow[\delta\to 0]{} -f_x(0) = -f(x)$

$\Rightarrow \Delta u(x) = -f(x)$. //